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EXPANSION PROCEDURES AND SIMILARITY LAWS FOR TRANSONIC FLOW

PART I. SLENDER BODIES AT ZERO INCIDENCE

J. D. Cole and A. F. Messiter

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AIR RESEARCH AND DEVELOPMENT COMMAND

GUGGENHEIM AERONAUTICAL LABORATORY

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1. Introduction.*

The purpose of this report is to provide a detailed and comprehensive account of a transonic approximation as applied to flows past wings and bodies. It is mainly concerned with the derivation of approximate equations, boundary conditions, etc., rather than with the more difficult problem of the solution of transonic flow problems. Thus the report contains for the most part a re-examination of the basic ideas, as presented for example, in Ref. 1. The essential new point of view introduced here is to regard the approximate transonic equations as part of a systematic expansion procedure. Thus, it becomes possible, in principle, to compute the higher terms of this approximation or at least to estimate errors.

In the next section the form of the expansion and the reasons for it are explained. In the succeeding sections the equations of motion, shock relations, and boundary conditions for the flow problem are presented and then the expansion procedure is applied systematically.

The resulting system of equations for the first, second, and higher approximations is presented in Section 5. The main results of interest for practical applications concern similarity laws and the pressure coefficient on the surface of slender bodies and these appear in Section 6. The remaining section treats bodies of non-circular cross-section.

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2. Motivation.

Linearized theory (for a complete account see the book by Ward, Ref. 2) is useful for estimating the pressure on slender bodies and wings in the subsonic and supersonic regimes. However, the results of the theory (cf. Ref. 2) predict an infinite pressure for bodies flying at the sonic velocity, Mach number $M_\infty = 1$. This result, so far removed from reality, indicates a failure in basic assumptions underlying the linearized theory. It also indicates that the linearized theory is a poor approximation for some range of Mach numbers about 1, the transonic regime. It is the aim of transonic theory to provide the proper mathematical description of the flow in this range and hence to allow calculation of pressures and overall forces.

From the point of view of mathematical expansion procedures the "breakdown" of linearized theory corresponds to a non-uniformity in the expansion procedure. The successive terms in the expansion, which are implicitly assumed to be small, become comparable with the first term (which is given by linearized theory).

In more detail, in a typical case there exists a velocity potential Φ which for steady flow past a given body depends on the dimensionless space coordinates (x, h) and the physical parameters δ, M_∞, γ:

δ = thickness ratio = t/L

t = maximum thickness of body

L = characteristic length, e.g. overall length of body

M_∞ = free stream Mach number = U/a_∞

U = flight speed

a_∞ = speed of sound at infinity

γ = ratio of specific heats

Φ is determined as satisfying a non-linear partial differential equation in (x, h) embodying conservation of mass, momentum, and energy, and certain boundary conditions. The body causing the disturbance to uniform flow is assumed to be thin so that the potential may be represented by the following type of expansion

$$\Phi(x,h; M_\infty, \delta) = UL\left\{ x + \epsilon_1(\delta)\phi_1(x,h;\beta) + \epsilon_2(\delta)\phi_2(x,h;\beta) + \cdots \right\} \tag{2-1}$$

where $\beta = \sqrt{|1-M_\infty^2|}$. The $\epsilon_i(\delta) \to 0$ as $\delta \to 0$ and $\frac{\epsilon_{i+1}}{\epsilon_i} \to 0$ as $\delta \to 0$. The first term in the expansion represents a uniform flow toward the body and the second term $\epsilon_1 \phi_1$ represents the potential of linearized theory. The remaining terms $\epsilon_2 \phi_2$, $\epsilon_3 \phi_3$ etc. represent second-order and third-order corrections etc. In any given case the ϵ_i are known functions of δ determined from the equation and boundary conditions. By requiring the non-linear equation for Φ to be satisfied identically for all orders of magnitude in δ, the linearized equation for ϕ_1 is found. Typically it is of the form

$$(1 - M_\infty^2)\,\phi_{1_{xx}} + \phi_{1_{hh}} + \cdots = 0 \tag{2-2}$$

In order for the expansion (2-1) to be a good approximation it is implicit that the velocity perturbations due to successive terms in the expansion are of successively smaller size over most of the flow field. Unfortunately this is not possible when $\beta \to 0$, $M_\infty \to 1$ because $\phi_{1_x} \to \infty$; thus the expansion is said to be non-uniform with respect to β. That is, for any given β, a sufficiently small $N(\beta)$ can be found so that for $\delta < N(\beta)$ the approximation is good*. But as $\beta \to 0$, $N \to 0$ also.

That such a non-uniformity might exist is evident from the mathematical structure of (2-2). For $M_\infty < 1$, ϕ_1 satisfies an elliptic differential equation similar to Laplace's equation, and for $M_\infty > 1$, ϕ_1 satisfies a hyperbolic equation similar to the wave equation. Due to the completely different dependence of solutions to these equations on the boundary conditions a non-uniformity might be expected. It is well known (cf. Ref. 3) that according to the exact equations a flow which is locally supersonic is described by a hyperbolic equation (has a wave structure) and flow which is locally subsonic is described by an elliptic equation. In the transonic range the velocity may change from subsonic to supersonic in the flow field. Linearized theory is too crude to describe this feature of the flow but one will require it of a transonic theory. This implies that the basic equation of the transonic approximation is one of changing type (elliptic to hyperbolic). This also implies that the equation is non-linear as the line across which the change of type takes place, the sonic line, may not be fixed in advance. Thus the transonic expansion must have a substantially different form from the "linearized" expansion (2-1) and in particular must involve different combinations of parameters M_∞, δ.

Before outlining the proposed expansion some physical aspects of the approximation will be discussed. Linearized theory is identical in content with acoustics; squares of disturbances are neglected (for a complete discussion cf. Ref. 2). In a system of coordinates fixed in the air at rest at infinity the flow is described as a system of acoustic waves

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*Of course, good has to be defined more precisely. For example, it might be demanded that the first omitted term $\epsilon_2 \phi_2$ amount to less than 10 per cent of $\epsilon_1 \phi_1$.

produced by the motion of the body. These acoustic waves travel with a constant speed a_∞ and each wave carries a small disturbance. When a body flies with a speed U close to a_∞ some of the waves emitted by the body are always close to the body and a build-up of pressure on the body may result. This result depends on the linearized description of the flow and causes the failure of approximations near $M_\infty = \frac{U}{a_\infty} = 1$. Actually, as these waves build up their interaction becomes important. The result is the famous steepening of the wave front described by Riemann (cf. Ref. 3, p. 45) in his theory of plane waves. In Riemann's theory it is shown that non-linear terms like $u u_x$ are responsible for the steepening of a plane wave traveling in the x-direction (u = flow velocity component in x-direction). From this point of view such a non-linearity is to be required in the basic equation of the transonic approximations. It is, in fact, the same non-linearity discussed in the last paragraph.

Another physically significant point concerns the relative orders of magnitude of perturbations from the uniform stream. It is implicit in the linearized theory that the (dimensionless) velocity perturbations in the x and r directions $\epsilon_1 \phi_{1x}$, $\epsilon_1 \phi_{1r}$ are of the same order of magnitude, namely ϵ_1. On the other hand an examination of the exact solution of a simple supersonic flow (cf. Ref. 3), the expansion around a corner, shows that the perturbations in the x and r directions (u, v) from a uniform sonic stream in the x-direction are related as $v \sim u^{3/2}$. A similar result holds for shock waves in the transonic range. This sort of order of magnitude relation will be required, in the large, for the perturbations of the basic transonic approximation.

If it is known that a potential exists (as will later be shown to be the case) this last requirement has the interesting consequence that the expansion must be carried out in a distorted set of coordinates. For example, represent the potential by $\epsilon_1(\delta)\,\phi_1(x,\tilde{h})$ where $\tilde{h}=\epsilon_1^{\frac{1}{2}}h$ so that the actual velocity perturbations $\frac{\partial}{\partial x}(\epsilon_1\phi_1)$, $\frac{\partial}{\partial h}(\epsilon_1\phi_1)$ are of the correct order of magnitude. A more vague interpretation of this coordinate $\tilde{h}$ is given by saying that in the transonic range perturbations caused by a body in the flow extend much farther in the direction transverse to the flow (large h) than in the flow direction and that a shrunken coordinate $\tilde{h}$ must be used to bring these disturbances back in view.

All of these physical considerations are keys to the more general form of expansion which must be used in the transonic case. The idea of the expansion is again that small disturbances in a uniform stream are caused by the introduction of a body of small thickness ratio δ and the expansion procedure is based on $\delta \to 0$. However now a more general form of expansion, allowing for different parameters and distorted coordinates must be used. Assuming a potential, a sufficiently general form is

$$\Phi(x,h;M_\infty,\delta,\gamma)=UL\left\{x+\epsilon_1(\delta)\phi_1(x,\tilde{h};P_1,P_2)+\epsilon_2(\delta)\phi_2(x,\tilde{h};P_1,P_2)+\cdots\right\} \tag{2-3}$$

where P_1, P_2 are functions of $(\gamma, M_\infty, \delta)$ which are independent of δ and $\tilde{h}=Ah$ where $A=A(\delta,M_\infty,\gamma)$.

The expansion parameters ϵ_1 , P_1 , P_2 and the coordinate distortion A must be determined in the course of finding the equations for

ϕ_1 . The main requirement to be imposed is that ϕ_1 satisfy a typical transonic equation. The choice of parameters in such an expansion is of course not unique but it will be shown that all such transonic expansions retain some typical features in common.

In the next section the basic equations of motion are presented and in the sections following that a general type of expansion procedure corresponding to (2-3) is applied.

3. Basic Equations of Motion and Boundary Conditions.

The basic equations of motion consist in statements of the conservation of mass, momentum, and energy in the form of differential equations and jump conditions across shock waves. To these must be added an equation of state and the Second Law of Thermodynamics. The medium is assumed to be a perfect, inviscid gas. The region under consideration always extends to infinity. The boundary conditions prescribe the flow at infinity and require the flow to be tangent to the body surface.

Assume that a characteristic length L exists, say, the body length and make the space coordinates dimensionless with L . Then the differential equations of motion can be written in terms of dimensionless coordinates as:

Continuity: $$\operatorname{div} \rho \vec{q} = 0 \qquad (3\text{-}1a)$$

Euler $$\vec{q} \cdot \nabla \vec{q} = \nabla \left(\frac{q^2}{2} \right) + \vec{\omega} \times \vec{q} = -\frac{1}{\rho} \nabla P \qquad (3\text{-}1b)$$

Entropy $$\vec{q} \cdot \nabla \left(\frac{P}{\rho^{\gamma}} \right) = 0 \qquad (3\text{-}1c)$$

System (3-1) together with the shock relations, which will not be written out (cf. Ref. 3 p. 51) comprise the basic system. An expansion procedure could be applied to the basic system; however here a modified system will be derived which contains some integrals of the basic system.

First note that (3-1c) implies that $\frac{p}{\rho^{\gamma}}$ is constant along streamlines, although of course the constant jumps across a shock wave. For all problems considered here, there will be a uniform state at infinity, so that (3-1c) implies

$$\frac{p}{\rho^{\gamma}} = \frac{p_\infty}{\rho_\infty^{\gamma}} e^{\frac{S-S_\infty}{c_v}} = k \quad \text{(say) is constant along a streamline} \tag{3-2}$$

The basic integral invariant is formed by taking the inner product of $\vec{q}$ and the Euler equation and integrating along a streamline. The result is

$$\frac{q^2}{2} + \int \frac{dp}{\rho} = \frac{q^2}{2} + \frac{a^2}{\gamma-1} = \quad \text{const. along streamlines} \tag{3-3}$$

Furthermore, the shock wave relations show that the constant in (3-3) is conserved across shock waves. Since a uniform state is assumed at upstream infinity the constant for each streamline is the same. Thus, the following invariant is obtained for the entire flow field

$$\frac{q^2}{2} + \frac{a^2}{\gamma-1} = \frac{U^2}{2} + \frac{a_\infty^2}{\gamma-1} = \frac{1}{2}\,\frac{\gamma+1}{\gamma-1}\,a^{*2} \tag{3-4}$$

By forming the gradient of the invariant (3-4) the vorticity laws are derived. Replacing $\nabla(q^2)$ from the Euler equation (3-1b) and using

(3-2) gives the vortex law.

$$\vec{\omega} \times \vec{q} = \frac{\rho^{\gamma-1}}{\gamma-1} \nabla k \tag{3-5}$$

Finally, the density may be eliminated from the continuity equation (3-1a) by evaluating $\vec{q} \cdot \nabla p$ using (3-2) and (3-1b). The result is

$$a^2 \operatorname{div} \vec{q} = \vec{q} \cdot \nabla\left(\frac{q^2}{2}\right) \tag{3-6}$$

Summarizing, the original system of differential equations can be replaced by (3-4), (3-5), and (3-6).

$$\left\{\begin{array}{lll} \text{Continuity} & a^2 \operatorname{div} \vec{q} = \vec{q} \cdot \nabla\left(\frac{q^2}{2}\right) & (3\text{-}6) \\ \text{Invariant} & \frac{q^2}{2} + \frac{a^2}{\gamma-1} = \frac{U^2}{2} + \frac{a_\infty^2}{\gamma-1} = \frac{1}{2}\frac{\gamma+1}{\gamma-1} a^{*2} & (3\text{-}4) \\ \text{Vorticity} & \vec{\omega} \times \vec{q} = \frac{\rho^{\gamma-1}}{\gamma-1} \nabla k & (3\text{-}5) \end{array}\right\} \quad (3\text{-}7)$$

and $\vec{q} \cdot \nabla k = 0$

In certain applications, for example in flow which is subsonic everywhere, the entropy and k are constant throughout. The vorticity law (3-5) then implies $\vec{\omega} = 0$ and a velocity potential exists. Then (3-6) becomes the single equation for the potential. It will be shown later that for the transonic flows considered this statement is true (to a certain order) even if shock waves are present.

It is convenient later to use certain relations deduced from the basic shock relation. These are the shock polar equation which relates the velocity components, and equations for the pressure jump, wave angle, and density jump.

The shock polar can be written

$$\text{Polar} \qquad v^{(1)^2} = \left(u^{(0)} - u^{(1)}\right)^2 \frac{u^{(0)}u^{(1)} - a^{*2}}{\frac{2}{\gamma+1} u^{(0)^2} - u^{(0)}u^{(1)} + a^{*2}} \tag{3-8a}$$

where superscript (0) denotes conditions upstream of the shock wave and superscript (1) denotes conditions downstream. $u^{(0)}$ is the magnitude of the velocity upstream. $u^{(1)}$ is the component of the velocity downstream in the direction of $u^{(0)}$ and $v^{(1)}$ is the component normal. u, v are related to the Cartesian components as follows

$$u^{(0)^2} = q_x^{(0)^2} + q_r^{(0)^2}$$

$$u^{(1)} = q_x^{(1)} \frac{q_x^{(0)}}{\sqrt{q_x^{(0)^2} + q_r^{(0)^2}}} + q_r^{(1)} \frac{q_r^{(0)}}{\sqrt{q_x^{(0)^2} + q_r^{(0)^2}}} \tag{3-8b}$$

$$v^{(1)} = -q_x^{(1)} \frac{q_r^{(0)}}{\sqrt{q_x^{(0)^2} + q_r^{(0)^2}}} + q_r^{(1)} \frac{q_x^{(0)}}{\sqrt{q_x^{(0)^2} + q_r^{(0)^2}}}$$

A special case of interest occurs when the shock is in the free stream so that $u^{(0)} = U$ and is in the x direction. For this case the following relations hold

Wave Angle $$\tan\theta_w = \frac{1 - q_x^{(1)}/U}{q_h^{(1)}/U} \tag{3-9}$$

Pressure $$P^{(1)} = P_\infty + \rho_\infty U(U - q_x^{(1)}) \tag{3-10}$$

Density $$\frac{\rho_\infty}{\rho^{(1)}} = \frac{q_x^{(1)}}{U} - \frac{q_h^{(1)}}{U\tan\theta_w} \tag{3-11}$$

In case the flow does not have axial-symmetry these formulas still hold if q_h is replaced by the resultant transverse component behind the shock.

The boundary condition to be applied at the body surface is one of tangent flow at the surface

$$\vec{q}\cdot\nabla S = 0 \qquad \text{where} \quad S(x,h,\theta) = 0 = h - \delta F(x,\theta) \tag{3-12}$$

is the equation of the body surface. Written out (3-12) is

$$\frac{q_h(x,\delta F,\theta)}{U} = \frac{q_x(x,\delta F,\theta)}{U}\,\delta\frac{\partial F}{\partial x} + \frac{q_\theta(x,\delta F,\theta)}{U}\,\frac{1}{F}\frac{\partial F}{\partial\theta} \tag{3-13a}$$

and for the special case of axial symmetry

$$\frac{q_h(x,\delta F)}{U} = \frac{q_x(x,\delta F)}{U}\,\delta F'(x) \qquad \text{where on the body} \quad h = \delta F(x) \tag{3-13b}$$

The other boundary condition is a statement that all flow disturbances die out at upstream infinity.

4. Deduction of the Form of the Expansion; Body of Revolution.

In this section it is shown how with the aid of certain general requirements the form of a transonic expansion can be deduced. The requirements are stated for a free stream Mach number of one $(M_\infty = 1)$. These requirements are:

(i) The shock wave relations should not degenerate in the expansion. Rather the shock waves should approximate real ones with jumps of velocity and pressure etc.

(ii) The differential equations of motion should be able to describe the flow satisfying boundary conditions on the body and at infinity. These two requirements define the expansion in terms of δ uniquely at $M_\infty = 1$. However the extension of the expansion for $M_\infty \neq 1$ is not unique, in a way which is shown in detail below. It is important to note that these requirements yield a transonic approximation which has all the qualitative features described in Section 2.

The shape of the body of revolution and the boundary condition on it are given by Eq. (3-13b). The expansion procedure is based on $\delta \to 0$ and it is assumed here that (M_∞, δ) are the parameters defining the flow. In accordance with the remarks made in Section 2 the following form of the expansion is assumed for the velocity components

$$\left\{ \begin{aligned} \frac{q_x(x,h;M_\infty,\delta)}{U} &= 1 + \epsilon_1(\delta)u_1(x,\tilde{h};P_1) + \epsilon_2(\delta)u_2(x,\tilde{h};P_1) + \cdots \\ \frac{q_h(x,h;M_\infty,\delta)}{U} &= \nu_1(\delta)V_1(x,\tilde{h};P_1) + \nu_2(\delta)V_2(x,\tilde{h};P_1) + \cdots \end{aligned} \right\} \qquad \text{(4-1a)}$$

where the ϵ_i and ν_i each form a decreasing sequence as $\delta \to 0$ and

$$\left\{ \begin{array}{l} \tilde{h} = \delta^{a} h \\ l_1 = \text{parameter independent of } \delta \end{array} \right\} \qquad (4\text{-}1b)$$

In accordance with requirement (i) the assumed form of the expansion (4-1a) is substituted into the shock polar relation as given by (3-8). Thus

$$\left\{ \begin{array}{l} \dfrac{u^{(o)}}{U} = 1 + \epsilon_1 u_1^{(o)} + \cdots \\ \dfrac{u^{(1)}}{U} = 1 + \epsilon_1 u_1^{(1)} + \cdots \\ \dfrac{v^{(1)}}{U} = \nu_1 \left(v_1^{(1)} - v_1^{(o)} \right) + \cdots \end{array} \right\} \qquad (4\text{-}2a)$$

and the shock polar, for $M_\infty = 1$, $a^* = U$, becomes

$$\nu_1^2 \left(v_1^{(1)} - v_1^{(o)} \right)^2 + \cdots = \epsilon_1^2 \left(u_1^{(o)2} - u_1^{(1)2} \right) \frac{1 + \epsilon_1 \left(u_1^{(o)} + u_1^{(1)} \right) + \cdots - 1}{\frac{2}{\gamma+1} \left(1 + \cdots \right) - 1 + \cdots + 1}$$

or

$$\nu_1^2 \left(v_1^{(1)} - v_1^{(o)} \right)^2 + \cdots = \epsilon_1^3 \left(u_1^{(o)} - u_1^{(1)} \right)^2 \left(u_1^{(o)} + u_1^{(1)} \right) \frac{\gamma+1}{2} + \cdots \qquad (4\text{-}2b)$$

In order to avoid a degenerate polar it is necessary that $\nu_1^2 \sim \epsilon_1^3$ as $\delta \to o$. Hence we take

$$\nu_1^2 = \epsilon_1^3 \qquad (4\text{-}3)$$

The assumed form of the expansion is now used in the modified continuity equation (3-6) and use is made of the invariant (3-4). Thus

$$\frac{q^2}{U^2} = 1 + 2\epsilon_1 u_1 + 2\epsilon_2 u_2 + \epsilon_1^2 u_1^2 + 2\epsilon_1\epsilon_2 u_1 u_2 + \epsilon_2^2 u_2^2 + \cdots + \epsilon_1^3 v_1^2 + \cdots$$

$$\frac{a^2}{U^2} = \frac{1}{M_\infty^2} + \frac{\gamma-1}{2}\left(1 - \frac{q^2}{U^2}\right) = \frac{1}{M_\infty^2} - (\gamma-1)\epsilon_1 u_1 - (\gamma-1)\epsilon_2 u_2 - \frac{\gamma-1}{2}\epsilon_1^2 u_1^2 \cdots$$

$$\frac{\vec{q}}{U}\cdot\nabla = (1 + \epsilon_1 u_1 + \epsilon_2 u_2 + \cdots)\frac{\partial}{\partial x} + \epsilon_1^{3/2}\delta^a v_1 \frac{\partial}{\partial \tilde{r}} + \cdots$$

and Eq. (3-6) becomes

$$\left(\frac{1}{M_\infty^2} - (\gamma-1)\epsilon_1 u_1 - (\gamma-1)\epsilon_2 u_2 - \frac{\gamma-1}{2}\epsilon_1^2 u_1^2 - \cdots\right)\Big(\epsilon_1 u_{1x} + \epsilon_2 u_{2x} + \cdots$$
$$+ \epsilon_1^{3/2}\delta^a\left(v_{1\tilde{r}} + \frac{1}{\tilde{r}}v_1\right) + \nu_2\delta^a\left(v_{2\tilde{r}} + \frac{1}{\tilde{r}}v_2\right) + \cdots\Big)$$

$$= \left((1 + \epsilon_1 u_1 + \epsilon_2 u_2 + \cdots)\frac{\partial}{\partial x} + \epsilon_1^{3/2}\delta^a v_1\frac{\partial}{\partial \tilde{r}} + \cdots\right)\Big(\epsilon_1 u_1 + \epsilon_2 u_2$$
$$+ \epsilon_1^2\frac{u_1^2}{2} + \epsilon_1\epsilon_2 u_1 u_2 + \frac{\epsilon_2^2}{2}u_2^2 + \cdots + \frac{\epsilon_1^3 v_1^2}{2} + \epsilon_1^{3/2}\nu_2 v_1 v_2 + \cdots\Big) \tag{4-4}$$

The form of Eq. (4-4) when $M_\infty = 1$ is now studied:

$$\left(1 - (\gamma-1)\epsilon_1 u_1 - (\gamma-1)\epsilon_2 u_2 - \frac{\gamma-1}{2}\epsilon_1^2 u_1^2 - \cdots\right)\Big(\epsilon_1 u_{1x} + \epsilon_2 u_{2x} + \cdots$$
$$+ \epsilon_1^{3/2}\delta^a\left(v_{1\tilde{r}} + \frac{1}{\tilde{r}}v_1\right) + \nu_2\delta^a\left(v_{2\tilde{r}} + \frac{1}{\tilde{r}}v_2\right) + \cdots\Big)$$

$$= \left((1 + \epsilon_1 u_1 + \epsilon_2 u_2 + \cdots)\frac{\partial}{\partial x} + \epsilon_1^{3/2}\delta^a v_1\frac{\partial}{\partial \tilde{r}} + \cdots\right)\Big(\epsilon_1 u_1 + \epsilon_2 u_2 + \epsilon_1^2\frac{u_1^2}{2}$$
$$+ \epsilon_1\epsilon_2 u_1 u_2 + \epsilon_2^2\frac{u_2^2}{2} + \cdots + \frac{\epsilon_1^3 v_1^2}{2} + \epsilon_1^{3/2}\nu_2 v_1 v_2 + \cdots\Big) \tag{4-5}$$

and the requirement is made that the largest terms in Eq. (4-5) yield an equation which contains the possibility of describing the flow field in the large. It is easily seen that the largest terms are

$$\epsilon_1^{3/2}\,\delta^a\left(V_{1_{\tilde{\lambda}}} + \frac{1}{\tilde{\lambda}}V_1\right) = \epsilon_1^2(\gamma+1)\,u_1 u_{1_\chi} \tag{4-6}$$

If either $\delta^a = o(\epsilon_1^{1/2})$ or $\epsilon_1^{1/2} = o(\delta^a)$ this contains only one of the unknowns and yields unreasonable relations. Therefore choose

$$\epsilon_1^{1/2} = \delta^a \tag{4-7a}$$

and obtain the first approximation equation for $M_\infty = 1$.

$$V_{1_{\tilde{\lambda}}} + \frac{1}{\tilde{\lambda}}V_1 = (\gamma+1)\,u_1 u_{1_\chi} \tag{4-7b}$$

The extension of this form of expansion to $M_\infty \neq 1$ can be performed in various ways by requiring the term $\epsilon_1\left(\frac{1}{M_\infty^2}-1\right)u_{1_\chi}$ in Eq. (4-4) to be of order ϵ_1^2 ; that is

$$\left(\frac{1}{M_\infty^2}-1\right)\epsilon_1 = \mathcal{O}(\epsilon_1^2) \qquad \text{as} \qquad \epsilon_1 \to 0 \tag{4-8}$$

Note that if $\epsilon_1^2 = o\left(\left[\frac{1}{M_\infty^2}-1\right]\epsilon_1\right)$ as $\epsilon_1 \to 0$ then a degenerate form of equation is obtained while if $\left(\frac{1}{M_\infty^2}-1\right)\epsilon_1 = o(\epsilon_1^2)$ a more restricted approximation scheme is obtained. The requirement of Eq. (4-8) can be satisfied in a variety of ways, of which the following simple one will be used in this section

$$\frac{1-M_\infty^2}{\epsilon_1} = P_1 \quad = \text{similarity parameter independent of } \delta = K \text{ (say)} \qquad (4\text{-}9)$$

As $\epsilon_1(\delta) \to 0$, $M_\infty(\delta)$ is taken to be such a function that P_1 is independent of δ; or

$$M_\infty^2 = 1 - \epsilon_1(\delta) K \qquad (4\text{-}10)$$

The use of this parameter also extends the shock polar (4-2b) away from $M_\infty = 1$. Thus, the largest terms in Eq. (4-4) yield the first approximation equation

$$K u_{1_x} + V_{1_{\tilde{r}}} + \frac{1}{\tilde{r}} V_1 = (\gamma+1) u_1 u_{1_x} \qquad (4\text{-}11)$$

Next, another relation between (ϵ_1, δ) is found from the boundary condition of tangent flow Eq. (3-13b). Using the assumed form of expansion (4-1a), Eq. (3-13b) becomes

$$\epsilon_1^{3/2} V_1(x, \delta^{a+1} F) + \nu_2 V_2(x, \delta^{a+1} F) + \cdots = \left(1 + \epsilon_1 u_1(x, \delta^{a+1} F) + \cdots\right) \delta F'(x) \qquad (4\text{-}12)$$

Eq. (4-12), which holds on the boundary gives another expansion as $\delta \to 0$, the "expansion on the boundary". If we assume $a > -1$, then the behavior of V_1, V_2 etc. as $\tilde{r} \to 0$ is decisive in fixing the form of this expansion. If the physically reasonable assumption is made that u_1 is not too singular, $O(\log \tilde{r})$ as $\tilde{r} \to 0$, then from Eq. (4-6) it is seen that V_1 may be $\sim \frac{1}{\tilde{r}}$. On physical grounds this type of behavior is to be expected as it is recognized as the fluid source line on the axis often used to represent a body. Thus we write, near the body,

$$V_1(x, \tilde{r}) \rightarrow \frac{S_1(x)}{\tilde{r}} \quad \text{as} \quad \tilde{r} \rightarrow 0 \qquad S_1 = \text{source strength} \tag{4-13}$$

Our later considerations will show that the case $a = -1$ can not lead to any transonic approximation, while $a < -1$ is no good for obvious reasons.

By using the limiting behavior expressed by Eq. (4-13) the expansion on the boundary (4-12) becomes

$$\epsilon_1^{3/2}\left(\frac{S_1(x)}{\delta^{a+1}F} + \cdots\right) + \cdots = \delta F'(x) \tag{4-14}$$

Equating the largest terms in (4-14) as $\delta \rightarrow 0$ shows that

$$\epsilon_1^{3/2} = \delta^{a+2} \tag{4-15}$$

and

$$S_1(x) = F(x)\,F'(x) \tag{4-16}$$

Eq. (4-16) is the conventional result for source strength in slender body theory, namely the source strength varies as the rate of change of the body cross-section area. (Cf. Ref. 4.)

Thus the parameters for the first approximation are determined from Eq. (4-3), (4-7a), and (4-15) to be

$$a = 1, \quad \epsilon_1 = \delta^2, \quad \nu_1 = \delta^3 \tag{4-17a}$$

and

$$K = \frac{1 - M_\infty^2}{\delta^2} \qquad \text{(4-17b)}$$

K is practically the similarity parameter introduced by Karman in Ref. 1.

To complete the specification of the problem for the first approximation the vorticity laws which depend on the entropy gradient must be considered. An estimate of the entropy changes can be obtained from the shock relations. It is sufficient for our purposes to consider the shock in the undisturbed stream. Similar results will hold in the general case. The expansion (4-1a) now reads

$$\begin{aligned} \frac{q_x}{U} &= 1 + \delta^2 u_1(x, \tilde{r}; K) + \epsilon_2(\delta) u_2(x, \tilde{r}; K) + \cdots \\ \frac{q_r}{U} &= \delta^3 v_1(x, \tilde{r}; K) + \eta_2(\delta) v_2(x, \tilde{r}; K) + \cdots \end{aligned} \qquad \tilde{r} = \delta r \qquad \text{(4-18)}$$

and using this in the expression (3-10) for the pressure ratio across the shock wave shows

$$\frac{P^{(1)}}{P_\infty} = 1 - \delta^2 \gamma u_1^{(1)} + \cdots \qquad \text{(4-19)}$$

Similarly expansions for the wave angle and density ratio can be obtained from Eqs. (3-9) and (3-11) respectively

$$\tan \theta_w = -\frac{1}{\delta} \frac{u_1^{(1)}}{v_1^{(1)}} + \cdots \qquad \text{(4-20a)}$$

Consistent with (4-20a) the shock shape can be written

$$\tilde{h} = g_1(x) + \cdots \qquad (4\text{-}20b)$$

he density ratio is

$$\frac{\rho_\infty}{\rho^{(1)}} = 1 + \delta^2 u_1^{(1)} + \cdots \qquad (4\text{-}21)$$

Superscript (1) means evaluation of the functions on the shock as given by (4-20b). Now

$$k^{(1)} = \frac{P^{(1)}}{\rho^{(1)\gamma}}$$

$$= \left(1 - \delta^2 \gamma u_1^{(1)} + \cdots\right)\left(1 + \delta^2 u_1^{(1)} + \cdots\right)^{\gamma} \frac{P_\infty}{\rho_\infty^{\gamma}}$$

$$\frac{k^{(1)}}{k_\infty} = 1 + o(\delta^2) \qquad (4\text{-}22)$$

That is, at the shock there is no entropy change of order δ^2, and since the entropy is constant on streamlines in the flow downstream of the shock a relation similar to (4-22) can be assumed to hold everywhere in the field. If there is more than one shock including some in a non-uniform stream the order relation specified by (4-22) still holds. Hence, in the entire flow

$$\frac{k}{k_\infty} = 1 + o(\delta^2) \qquad (4\text{-}23)$$

Using this result in the vorticity law (3-5), the flow can be seen to be irrotational in the first approximation

$$-\frac{q_x}{U}\left(\frac{1}{U}\frac{\partial q_x}{\partial r}-\frac{1}{U}\frac{\partial q_r}{\partial x}\right)=\frac{1}{U^2}\frac{\rho^{\gamma-1}}{\gamma-1}\frac{\partial k}{\partial r}$$

$$-\left(1+\delta^2 u_1+\dots\right)\left(\delta^3\left(u_{1_{\tilde r}}-v_{1_x}\right)-\dots\right)=\frac{1}{\gamma-1}\frac{\rho_\infty}{\left(1+\delta^2 u_1+\dots\right)^{\gamma-1}}\frac{k_\infty}{U^2}\,\delta\frac{\partial}{\partial\tilde r}\left(O(\delta^2)\right)$$

so that

$$u_{1_{\tilde r}}-v_{1_x}=0 \tag{4-24}$$

This completes the specification of the problem for u_1, v_1 and the first approximation.

To determine the further course of the expansion certain steps must be repeated for the next order terms. Returning to the modified continuity equation the second order of terms in (4-5) might be

$$\nu_2\,\delta\left(v_{2_{\tilde r}}+\frac{1}{\tilde r}v_2\right)=(\gamma+1)\delta^2\epsilon_2\left(u_1u_2\right)_x+\delta^6 v_1\left(u_{1_{\tilde r}}+v_{1_x}\right)$$
$$+\delta^6\frac{\gamma+1}{2}u_1^2u_{1_x}+\delta^6(\gamma-1)u_1\left(v_{1_{\tilde r}}+\frac{1}{\tilde r}v_1\right) \tag{4-25}$$

Reasoning as before equations capable of satisfying the boundary condition at the body and infinity will be found if

$$\nu_2=\epsilon_2\,\delta \tag{4-26}$$

The extension of the equation for $M_\infty\neq 1$ follows automatically now by $M_\infty^2 = 1 - K\delta^2$. We have, as a possibility,

$$\epsilon_2 \delta^2 K u_{2_x} + \nu_2 \delta \left(v_{2_{\tilde{r}}} + \frac{1}{\tilde{r}} v_2 \right) = (\gamma+1)\delta \epsilon_2^2 (u_1 u_2)_x + O(\delta^6) \tag{4-27}$$

The second relation between ϵ_2 and ν_2 can be found from the expansion of the boundary condition on the boundary. To carry this out it is necessary to know the first few terms of the expansions for u_1, v_1. Introducing a potential $\varphi_1(x, \tilde{r}; K)$ to satisfy Eq. (4-24) the Eq. (4-11) determines φ_1

$$\varphi_{1_{\tilde{r}\tilde{r}}} + \frac{1}{\tilde{r}} \varphi_{1_{\tilde{r}}} = \left((\gamma+1)\varphi_{1_x} - K \right) \varphi_{1_{xx}} \tag{4-28}$$

The beginning of the expansion of φ_1 near $\tilde{r} = 0$ and near the body has already been determined in Eq. (4-13). The first two terms $\varphi_1 = S_1(x) \log \tilde{r} + g_1(x) + \cdots$ of the expansion are solutions to Eq. (4-28) with the right hand side equal to zero. $S_1(x)$ is known but $g_1(x)$ is an unknown function of x. The successive terms in the expansion can be obtained by iteration using the first terms in the right hand side. The result is the following asymptotic expansion near the axis.

$$\varphi_1(x, \tilde{r}; K) = S_1(x) \log \tilde{r} + g_1(x; K) + \tilde{r}^2 \log^2 \tilde{r} \cdot \left(\frac{\gamma+1}{4} S_1' S_1'' \right) + O(\tilde{r}^2 \log \tilde{r})$$

$$v_1(x, \tilde{r}; K) = \frac{S_1(x)}{\tilde{r}} + O(\tilde{r} \log^2 \tilde{r}) \tag{4-29}$$

Now two possibilities with respect to Eq. (4-27) need to be investigated, $\epsilon_2 = \delta^4$ or $\delta^4 \doteq O(\epsilon_2)$. In the latter case the terms $O(\delta^6)$ drop out of Eq. (4-27) to give

$$K u_{2_x} + v_{2_{\tilde{r}}} + \frac{1}{\tilde{r}} v_2 = (\gamma+1)(u_1 u_2)_x \tag{4-30}$$

Then, applying the same reasoning as before

$$v_2(x,\tilde{r}) \rightarrow \frac{S_2(x)}{\tilde{r}} + \cdots \qquad \text{as} \qquad \tilde{r} \rightarrow 0 \tag{4-31}$$

and ν_2 can be found from the expansion of the boundary condition (4-12):

$$\delta^3\left(\frac{S_1(x)}{\delta^2 F} + O(\delta^2 \log^2 \delta)\right) + \nu_2\left(\frac{S_2(x)}{\delta^2 F} + \cdots\right) = \left(1 + 2\delta^2 S_1' \log\delta + \cdots\right)\delta F'(x) \tag{4-32}$$

The second order terms in this expansion are thus $O(\delta^3 \log\delta)$ and the boundary condition can be satisfied if

$$\nu_2 = \delta^5 \log\delta \tag{4-33a}$$

and, from (4-26),

$$\epsilon_2 = \delta^4 \log\delta \tag{4-33b}$$

This result is consistent with the assumption made that $\delta^4 \neq O(\epsilon_2)$. If on the other hand the assumption is made that $\epsilon_2 = \delta^4$, $v_2 \sim \frac{1}{\tilde{r}} \log\tilde{r}$ as $\tilde{r} \rightarrow 0$ instead of as shown in (4-31). This follows from the behavior as $\tilde{r} \rightarrow 0$ of the terms of $O(\delta^6)$ in (4-27). No consistent expansion can be found this way so that the possibility $\epsilon_2 = \delta^4$ is ruled out. As the procedure continues the terms which were left out appear in the equation for

(u_3, v_3). In fact σ_2 can not be found until they are considered.

The form of the expansion up to the second step is thus determined and the vorticity laws provide the second equation as before. The shock relations and the entropy jump across the shock must be considered first. A consistent way to treat the shock waves is by satisfying the shock conditions on the shock front $\tilde{r} = \sigma_1(x)$ as found in the first approximation. In all transonic problems involving a shock wave the position of the shock cannot be fixed in advance and the determination of $\sigma_1(x)$ is part of the solution of the first approximation. Some details will now be given.

The wave angle equation (3-9) indicates

$$\tan \Theta_w = -\frac{1}{\delta}\frac{u^{(1)}}{v^{(1)}} + O(\delta \log \delta) \tag{4-34}$$

in view of the second terms in the expansion. This implies a shock shape of the form

$$\tilde{r} = \sigma_1(x) + \delta^2 \log \delta \; \sigma_2(x) + \cdots \tag{4-35}$$

The velocity components behind the shock can be expanded as

$$\frac{q_x^{(1)}}{U} = 1 + \delta^2 u_1(x, \sigma_1 + \delta^2 \log \delta\, \sigma_2 + \cdots) + \delta^4 \log \delta\; u_2(x, \sigma_1 + \cdots) + \cdots$$

$$\frac{q_x^{(1)}}{U} = 1 + \delta^2 u_1(x, \sigma_1) + \delta^4 \log \delta \left(u_2(x, \sigma_1) + \sigma_2\, u_{1_{\tilde{r}}}(x, \sigma_1) \right) + \cdots \tag{4-36a}$$

$$\frac{q_r^{(1)}}{U} = \delta^3 v_1(x, \sigma_1) + \delta^5 \log \delta \left(v_2(x, \sigma_1) + \sigma_2\, v_{1_{\tilde{r}}}(x, \sigma_1) \right) + \cdots \tag{4-36b}$$

Now, the pressure and density relations (3-10) and (3-11) become to the second order

$$\frac{p^{(1)}}{p_\infty} = 1 - \delta^2 \gamma u_1(x,\sigma_1) - \delta^4 \log\delta\, \gamma\left(u_2(x,\sigma_1) + \sigma_2 u_{1_{\tilde{\lambda}}}(x,\sigma_1)\right) + \cdots \tag{4-37}$$

$$\frac{\rho_\infty}{\rho^{(1)}} = 1 + \delta^2 u_1(x,\sigma_1) + \delta^4 \log\delta \left(u_2(x,\sigma_1) + \sigma_2 u_{1_{\tilde{\lambda}}}(x,\sigma_1)\right) + \cdots \tag{4-38}$$

Thus the entropy change across the shock is specified by

$$\frac{k^{(1)}}{k_\infty} = 1 + o(\delta^4 \log\delta) \tag{4-39}$$

Again, consideration of the vorticity law (3-5) shows that the flow is irrotational to the second order

$$u_{2_{\tilde{\lambda}}} - v_{2_x} = 0 \tag{4-40}$$

The formula for the shock polar to the second order can also be worked out.

The procedure just outlined is thus complete and can be carried out to any order. The results about the entropy jump across shock waves which were used in this section were derived for points off the boundary. Similar reasoning can be applied to shocks on the boundary to show that the flow is effectively isentropic there.

5. Summary of Equations Derived in the Expansion.

Continuing the procedure started in the previous section leads to the following expansion

$$\frac{q_x}{U} = 1 + \delta^2 u_1(x,\tilde{r};K) + \delta^4 \log\delta\, u_2(x,\tilde{r};K) + \delta^4 u_3(x,\tilde{r};K) + O(\delta^6 \log^2\delta) \tag{5-1a}$$

$$\frac{q_r}{U} = \delta^3 v_1(x,\tilde{r};K) + \delta^5 \log\delta\, v_2(x,\tilde{r};K) + \delta^5 v_3(x,\tilde{r};K) + O(\delta^7 \log^2\delta) \tag{5-1b}$$

$$\tilde{r} = \delta r \quad , \quad K = \frac{1-M_\infty^2}{\delta^2}$$

$$\left\{ \begin{aligned} K u_{1x} + v_{1\tilde{r}} + \frac{1}{\tilde{r}} v_1 &= (\gamma+1)\, u_1 u_{1x} \\ u_{1\tilde{r}} - v_{1x} &= 0 \end{aligned} \right\} \qquad \begin{aligned} &(5\text{-}2a) \\ &(5\text{-}2b) \end{aligned}$$

$$\left\{ \begin{aligned} K u_{2x} + v_{2\tilde{r}} + \frac{1}{\tilde{r}} v_2 &= (\gamma+1)(u_1 u_2)_x \\ u_{2\tilde{r}} - v_{2x} &= 0 \end{aligned} \right\} \qquad \begin{aligned} &(5\text{-}3a) \\ &(5\text{-}3b) \end{aligned}$$

$$\left\{ \begin{aligned} K u_{3x} + v_{3\tilde{r}} + \frac{1}{\tilde{r}} v_3 &= (\gamma+1)(u_1 u_3)_x + v_1(u_{1\tilde{r}} + v_{1x}) - 2\gamma K u_1 u_{1x} \\ &\quad + \frac{(2\gamma-1)(\gamma+1)}{2} u_1^2 u_{1x} \\ u_{3\tilde{r}} - v_{3x} &= 0 \end{aligned} \right\} \qquad \begin{aligned} &(5\text{-}4a) \\ &(5\text{-}4b) \end{aligned}$$

$$\frac{k(x,\tilde{r};K)}{k_\infty} = 1 + O(\delta^6 \log^2\delta) \tag{5-5}$$

That is, the flow is irrotational in the first three orders. Potentials $\varphi_1, \varphi_2, \varphi_3$ can be introduced:

$$K \varphi_{1xx} + \varphi_{1\tilde{r}\tilde{r}} + \frac{1}{\tilde{r}} \varphi_{1\tilde{r}} = (\gamma+1)\varphi_{1x}\varphi_{1xx} \tag{5-6}$$

$$K \varphi_{2xx} + \varphi_{2\tilde{r}\tilde{r}} + \frac{1}{\tilde{r}} \varphi_{2\tilde{r}} = (\gamma+1)(\varphi_{1x}\varphi_{2x})_x \tag{5-7}$$

$$K\varphi_{3_{xx}} + \varphi_{3_{\tilde{r}\tilde{r}}} + \frac{1}{\tilde{r}}\varphi_{3_{\tilde{r}}} = (\gamma+1)(\varphi_{1_x}\varphi_{3_x})_x + 2\varphi_{1_{\tilde{r}}}\varphi_{1_{x\tilde{r}}}$$
$$- 2\gamma K\varphi_{1_x}\varphi_{1_{xx}} + \frac{(2\gamma-1)(\gamma+1)}{2}\varphi_{1_x}^{\ 2}\varphi_{1_{xx}} \tag{5-8}$$

The boundary conditions on the body for these φ's are replaced by the expansion near the axis as carried out in Eq. (4-29). Further terms in the asymptotic expansions can be found as outlined in the previous section and the results are

$$\varphi_1(x,\tilde{r};K) = S_1(x)\log\tilde{r} + g_1(x;K) + \tilde{r}^2\log^2\tilde{r}\left(\frac{\gamma+1}{4}S_1'S_1''\right)$$
$$+ \frac{\tilde{r}^2\log\tilde{r}}{4}\left((\gamma+1)(S_1'g_1')' - 2(\gamma+1)S_1'S_1'' - KS_1''\right)$$
$$+ \frac{\tilde{r}^2}{4}\left(\frac{3}{2}(\gamma+1)S_1'S_1'' - (\gamma+1)(S_1'g_1')' + KS_1'' + (\gamma+1)g_1'g_1'' - g_1''K\right)$$
$$+ O(\tilde{r}^4\log^3\tilde{r}) \text{ as } \tilde{r}\to 0 \text{ near the body} \quad 0<x<1 \tag{5-9}$$

$$\varphi_2(x,\tilde{r};K) = S_2(x)\log\tilde{r} + g_2(x;K) + \frac{\gamma+1}{4}(S_1S_2')'\tilde{r}^2\log^2\tilde{r} + O(\tilde{r}^2\log\tilde{r}) \tag{5-10}$$

$$\varphi_3(x,\tilde{r};K) = S_1S_1'\log^2\tilde{r} + O(\log\tilde{r}) \tag{5-11}$$

where the source strengths S_1, S_2 are known from the body shape

$$S_1(x) = F(x)F'(x) \tag{5-12a}$$

$$S_2(x) = -2S_1S_1' = -2FF'(FF')' \tag{5-12b}$$

and the g_1, g_2 are unknown functions. In general, to find g_1, g_2 the differential equations must be solved and as will be seen in more detail later these functions are important for the pressure distribution. It is to be noted that in the expansion on the boundary the largest terms due to φ_2 and φ_3 are of the same order. However it can be shown that the largest term due to u_4, v_4 in the expansion on the boundary is of higher order than those due to φ_2, φ_3 so that the expansion can be broken off.

The differential equation (5-6) for φ_1 is the classical transonic equation. It is non-linear and of changing type. The equation for φ_2 is however linear with variable coefficients depending on φ_1. The equation for φ_3 and all the succeeding equations are also linear with variable coefficients and they may have various known forcing terms on the right hand side.

6. Surface Pressures, Drag and Similarity.

The calculation of surface pressures depends on the velocity components on the boundary $\tilde{r} = \delta^2 F$. Using the expansions (5-9), (5-10), and (5-11), together with the expressions for the source strengths (5-12) an expansion for the velocity components on the boundary can be written

$$\frac{q_x(x, \delta F)}{U} = 1 + \delta^2 \log\delta \,(2S_1') + \delta^2 (S_1' \log F + g_1') + O(\delta^4 \log^2\delta) \tag{6-1a}$$

$$\frac{q_n(x, \delta F)}{U} = \delta F' + O(\delta^3 \log\delta) \tag{6-1b}$$

Next, the relation

$$\frac{k}{k_\infty} = \frac{p}{p_\infty}\left(\frac{\rho_\infty}{\rho}\right)^{\gamma} = 1 + o(\delta^2) \tag{6-2}$$

which holds throughout is solved for $\frac{p}{p_\infty}$:

$$\frac{p}{p_\infty} = \left(\frac{a^2}{U^2} M_\infty^2\right)^{\frac{\gamma}{\gamma-1}} \left(1 + o(\delta^2)\right) \tag{6-3}$$

An expansion of $\frac{a^2}{U^2}$ on the boundary can be found from the invariant (3-4) by using (6-1):

$$\frac{a^2}{U^2} = 1 - \delta^2 \log\delta \left(2(\gamma-1)S_1'\right) + \delta^2\left(K - (\gamma-1)\left[S_1'\log F + g_1' + \frac{S_1^2}{2F^2}\right]\right) + \Theta(\delta^4 \log^2\delta) \tag{6-4}$$

Thus, on the boundary

$$\frac{p}{p_\infty} = 1 - \delta^2\log\delta\left(2\gamma S_1'\right) - \delta^2\gamma\left(S_1'\log F + g_1' + \frac{S_1^2}{2F^2}\right) + \Theta(\delta^4\log^2\delta) \tag{6-5}$$

and

$$C_p = \frac{p - p_\infty}{\frac{\rho_\infty}{2}U^2} = \frac{\frac{p}{p_\infty} - 1}{\frac{\gamma}{2}M_\infty^2} = -\delta^2\log\delta\left(4S_1'\right) - \delta^2\, 2\left(S_1'\log F + g_1' + \frac{S_1^2}{2F^2}\right) + \Theta(\delta^4\log^2\delta) \tag{6-6}$$

Eq. (6-6) expresses the pressure on the body in terms of the body shape function $F(x)$, a source strength $S_1 = FF'$, and an unknown function $g_1(x)$. $g_1(x)$ has to be found from the solution of a boundary value problem for φ_1 . Thus g_1 depends on K, $g_1 = g_1(x;K)$. The dominant term $\Theta(\delta^2\log\delta)$ in (6-6) is part of the contribution of the sources along the axis (the other part is $\delta^2 S_1 \log F$). A corresponding term due to sources exists in linearized supersonic theory where the expansion starts (cf. Ref. 4, p. 89)

$$C_p = -\delta^2 \log \delta \, (2S_1') + \mathcal{O}(\delta^2) \qquad \text{(linearized subsonic or supersonic)} \quad (6\text{-}7)$$

Thus, the sources contribute a dominant term in transonic flow which is twice as large as that in supersonic or subsonic flow. From a practical point of view, however, the term $\mathcal{O}(\delta^2)$ is appreciable in comparison with the first term.

The term $g_1(x; K)$ represents the effect of the non-linear interaction between sources and is the essential transonic part of the problem. The result is emphasized by a first-order similarity rule for the pressure coefficient. Define a quantity $\tilde{C}_p$ by

$$\tilde{C}_p = \frac{C_p}{\delta^2} + 2S_1' \log \delta^2 F + \frac{S_1^2}{F^2} = -2g_1'(x; K) + \mathcal{O}(\delta^2 \log^2 \delta) \qquad (6\text{-}8)$$

The first order similarity rule is

$$\tilde{C}_p = \tilde{C}_p(x; K) \qquad \text{with an error } \mathcal{O}(\delta^2 \log^2 \delta) \quad (6\text{-}9)$$

The rule states that for bodies with similar shapes, the effect of the sources can be subtracted out so that there is a transonic similarity for the non-linear interaction effects. The rule in a form equivalent to (6-8) was given by Oswatitsch and Berndt (Ref. 5). An equivalent form of the rule can be written as

$$\tilde{C}_{p_E} = \frac{C_p}{\delta^2} + 4S_1' \log \delta = fn(x; K) + \mathcal{O}(\delta^2 \log^2 \delta) \qquad (6\text{-}10)$$

Corresponding to the similarity law for C_p given by Eq. (6-9) a similarity law for the drag coefficient can be given. Defining

$$C_D = \frac{2\pi\delta^2}{A_M}\int_0^1 C_p\, FF'\,dx \tag{6-11}$$

where

A_M=dimensionless maximum cross-sectional area = $\dfrac{\pi\delta^2}{4}$

it is found that for a body with a pointed nose and blunt base

$$\tilde{C}_D = \frac{C_D}{\delta^2} + 8F^2(1)F'^2(1)\log\delta^2F(1) = -16\int_0^1 FF'g_1'\,dx = f_1(K) + O(\delta^2\log^2\delta) \tag{6-12a}$$

or

$$\tilde{C}_D = -8\int_0^1 \tilde{C}_p\, FF'\,dx \tag{6-12b}$$

The pressure on the base has arbitrarily been put equal to P_∞ in this derivation.

7. Drag of Bodies of Nearly Circular Cross-Section.

The calculation of drag may be carried out in an analogous manner for a body whose cross-section is not quite circular. The surface of such a body will be represented by

$$S(x,r,\theta) = 0 = r - \delta F(x)\sqrt{1+\tau G(x,\theta)} \tag{7-1}$$

where

$$\tau = O(1)$$

and δ is again the thickness ratio of the body of revolution $r = \delta F(x)$.
If $G(x,\theta)$ is chosen so that

$$\int_0^{2\pi} G(x,\theta)\, d\theta = 0 \qquad (7\text{-}2)$$

then the distribution of cross-sectional area is the same as for the body of revolution. In particular, the maximum cross-section area $A_M = \frac{\pi}{4}\delta^2$, and this relation might be used as the definition of δ. The parameter τ is considered to be independent of δ, and no assumption is made concerning the relative magnitudes of the two quantities.

Since it has been shown that the flow over a body of revolution is approximately irrotational (Eq. (5-5)), it would be expected that the initial terms of the velocity expansions for the present case may also be represented by means of potential functions. A new radial coordinate $r^* = \frac{r}{\delta}$ is introduced, and the following form of expansion is assumed:

$$\frac{\Phi(x,r,\theta;M_\infty,\delta,\tau)}{UL} = x + \delta^2 \varphi_1(x,\tilde{r};K) + \delta^4 \log\delta\, \varphi_2(x,\tilde{r};K) + \cdots$$

$$+ \mu_1(\tau,\delta)\, \varphi_1^*(x,r^*,\theta;K) + \mu_2(\tau,\delta)\, \varphi_2^*(x,r^*,\theta;K) + \cdots \qquad (7\text{-}3)$$

where the functions $\varphi_i(x,\tilde{r};K)$ are the solutions, assumed known, for $\tau = 0$.

The boundary condition at the surface is obtained from Eq. (3-13a) if

$F(x,\theta)$ is replaced by $F(x)\sqrt{1+\tau G(x,\theta)}$. Substituting the expansion (7-3),

$$\delta^3 \varphi_{1\tilde{h}}\left(x, \delta^2 F\sqrt{1+\tau G}\right) + \delta^5 \log\delta\, \varphi_{2\tilde{h}}\left(x, \delta^2 F\sqrt{1+\tau G}\right) + \cdots$$
$$+ \frac{\mu_1}{\delta}\varphi^*_{1h^*}\left(x, F\sqrt{1+\tau G}, \theta\right) + \frac{\mu_2}{\delta}\varphi^*_{2h^*}\left(x, F\sqrt{1+\tau G}, \theta\right) + \cdots$$
$$= \frac{\delta\left(FF' + \frac{1}{2}\tau(F^2G)_x\right)}{F\sqrt{1+\tau G}}\left(1 + \delta^2\varphi_{1x}\left(x, \delta^2 F\sqrt{1+\tau G}\right) + \cdots + \mu_1\varphi^*_{1x}\left(x, F\sqrt{1+\tau G}, \theta\right) + \cdots\right)$$
$$+ \frac{\tau G_\theta}{2\delta F(1+\tau G)^{3/2}}\left(\mu_1 \varphi^*_{1\theta}\left(x, F\sqrt{1+\tau G}, \theta\right) + \cdots\right) \qquad (7\text{-}4)$$

Replacing the functions $\varphi_i\ (x,\tilde{h})$ by their known expansions near the axis, and performing Taylor expansions about $h^* = F$,

$$\delta\frac{S_1}{F}\left(1 - \frac{1}{2}\tau G + \frac{3}{8}\tau^2 G^2 + \cdots\right) + \delta^3\log\delta\ \frac{2S_1S_1'}{F}\left(1 - \frac{1}{2}\tau G + \cdots\right) + \cdots$$
$$+ \frac{\mu_1}{\delta}\left(\varphi^*_{1h^*}(x,F,\theta) + \frac{\tau F G}{2}\varphi^*_{1h^*h^*}(x,F,\theta) + \cdots\right) + \frac{\mu_2}{\delta}\left(\varphi^*_{2h^*}(x,F,\theta) + \cdots\right)$$
$$= \frac{\delta}{F}\left(FF' + \frac{1}{2}\tau(F^2G)_x\right)\left(1 - \frac{1}{2}\tau G + \frac{3}{8}\tau^2G^2 + \cdots\right)\left(1 + \delta^2\log\delta\ 2S_1' + \cdots\right)$$
$$+ \frac{\tau}{2\delta}\,\frac{G_\theta}{F}\left(1 + \cdots\right)\left(\mu_1\varphi^*_{1\theta}(x,F,\theta) + \cdots\right) \qquad (7\text{-}5)$$

Terms independent of τ cancel because of the definition of $\varphi_i(x,\tilde{h})$. The largest remaining terms determine μ_1 and the boundary condition for $\varphi_1^*(x,h^*,\theta)$:

$$\mu_1 = \tau\delta^2 \qquad (7\text{-}6a)$$

$$\varphi^*_{1_{r^*}}(x, F, \theta) = \frac{1}{2F}(F^2 G)_x \tag{7-6b}$$

The expansion has been chosen so that boundary conditions are applied on the mean surface F . Now, depending on the magnitude of τ , μ_2 will be either $\tau\delta^4 \log\delta$ or $\tau^2\delta^2$. In the discussion of Eq. (7-16) it will be pointed out that terms linear in τ cannot contribute to the drag, so we will take

$$\mu_2 = \tau^2\delta^2 \tag{7-7a}$$

$$\varphi^*_{2_{r^*}}(x, F, \theta) = -\frac{G}{4F}(F^2G)_x - \frac{FG}{2}\varphi^*_{1_{r^*r^*}}(x, F, \theta) + \frac{G_\theta}{2F}\varphi^*_{1_\theta}(x, F, \theta) \tag{7-7b}$$

without necessarily requiring that $\tau\delta^4 \log\delta < \tau^2\delta^2$.

The differential equation to be satisfied as essentially the same as Eq. (4-4):

$$\Big(1 + K\delta^2 + \cdots - (\gamma-1)\delta^2\varphi_{1_x} + \cdots - (\gamma-1)\tau\delta^2\varphi^*_{1_x} + \cdots\Big)\Big(K\delta^2\varphi_{1_{xx}} + \delta^4\big(\varphi_{1_{\tilde r\tilde r}} + \frac{1}{\tilde r}\varphi_{1_{\tilde r}}\big) + \cdots + K\tau\delta^2\varphi^*_{1_{xx}} + \tau\big(\varphi^*_{1_{r^*r^*}} + \frac{1}{r^*}\varphi^*_{1_{r^*}} + \frac{1}{r^{*2}}\varphi^*_{1_{\theta\theta}}\big) + \cdots\Big)$$

$$= \Big(\big(1 + \delta^2\varphi_{1_x} + \cdots + \tau\delta^2\varphi^*_{1_x} + \cdots\big)\frac{\partial}{\partial x} + \big(\delta^3\varphi_{1_{\tilde r}} + \cdots + \tau\delta\varphi^*_{1_{r^*}}\big)\frac{1}{\delta}\frac{\partial}{\partial r^*} + \frac{1}{r^{*2}}\big(\tau\varphi^*_{1_\theta} + \cdots\big)\frac{\partial}{\partial\theta}\Big)\Big(\delta^2\varphi_{1_x} + \cdots + \tau\delta^2\varphi^*_{1_x} + \cdots\Big) \tag{7-8}$$

The equations for $\varphi^*_i(x, r^*, \theta)$ are obtained by requiring that (7-8) be satisfied to every order of magnitude in the neighborhood of the body. It can be seen that the choice of μ_1 and μ_2 is not influenced by the differential equation, and that φ_1^* and φ_2^* must satisfy

$$\varphi^*_{i_{r^*r^*}} + \frac{1}{r^*}\varphi^*_{i_{r^*}} + \frac{1}{r^{*2}}\varphi^*_{i_{\theta\theta}} = 0 \qquad i = 1,2 \tag{7-9}$$

Expanding $G(x,\theta)$ in a Fourier series,

$$G(x,\theta) = \sum_{n=-\infty}^{\infty}{}' G_n(x)\, e^{in\theta} \tag{7-10}$$

where $G_0(x) = 0$ and $G_{-n}(x)$ = complex conjugate of $\overline{G_n(x)}$. Substitution of this expression in the boundary condition (7-6b) allows solution for $\varphi_1^*(x, r^*, \theta)$.

$$\varphi^*_{1_{r^*}}(x, F, \theta) = \frac{1}{2F}\sum_{n=-\infty}^{\infty}{}' \left(F^2 G_n\right)' e^{in\theta}$$

$$\varphi_1^*(x, r^*, \theta) = -\frac{1}{2}\sum_{n=-\infty}^{\infty}{}' \frac{F^{|n|}}{|n|\, r^{*|n|}}\left(F^2 G_n\right)' e^{in\theta} \tag{7-11}$$

The boundary condition (7-7b) becomes

$$\varphi^*_{2_{r^*}}(x, F, \theta) = \frac{1}{4F}\left(\sum_{n=-\infty}^{\infty}{}' G_n e^{in\theta}\right)\left(\sum_{n=-\infty}^{\infty}{}' |n|\left(F^2 G_n\right)' e^{in\theta}\right)$$

$$+ \frac{1}{4F}\left(\sum_{n=-\infty}^{\infty}{}' n\, G_n e^{in\theta}\right)\left(\sum_{n=-\infty}^{\infty}{}' \frac{n}{|n|}\left(F^2 G_n\right)' e^{in\theta}\right)$$

and the solution for $\varphi_2^*(x, r^*, \theta)$ is

$$\varphi_2^*(x, r^*, \theta) = -\frac{1}{4}\sum_{n=-\infty}^{\infty}{}' C_n(x)\,\frac{1}{|n|}\,\frac{F^{|n|}}{r^{*|n|}}\, e^{in\theta} \tag{7-12}$$

where $C_k(x)$ is defined by

$$\sum_{k=-\infty}^{\infty}{}' C_k(x) e^{ik\theta} = \sum_{n=-\infty}^{\infty}{}' \sum_{m=-\infty}^{\infty}{}' \Big(G_n |m| (F^2 G_m)' e^{i(n+m)\theta} + n G_n \frac{m}{|m|} (F^2 G_m)' e^{i(n+m)\theta} \Big)$$

with $C_0(x)$ taken equal to zero so that the solution dies out as $h^* \to \infty$.

The pressure is obtained by expanding Eq. (6-3), with the use of the invariant (3-4). At the body surface the result is

$$C_p = -2\Big(\delta^2 \varphi_{1x}(x, \delta^2 F\sqrt{1+\tau G}) + \cdots + \tau\delta^2 \varphi^*_{1x}(x, F\sqrt{1+\tau G}, \theta) + \tau^2\delta^2 \varphi^*_{2x}(x, F\sqrt{1+\tau G}, \theta) + \cdots\Big)$$
$$- \Big(\delta^3 \varphi_{1\tilde{h}}(x, \delta^2 F\sqrt{1+\tau G}) + \cdots + \tau\delta^2 \varphi^*_{1h^*}(x, F\sqrt{1+\tau G}, \theta) + \tau^2\delta^2 \varphi^*_{2h^*}(x, F\sqrt{1+\tau G}, \theta) + \cdots\Big)^2$$
$$- \frac{1}{\delta^4 F^2(1+\tau G)}\Big(\tau\delta^2 \varphi^*_{1\theta}(x, F\sqrt{1+\tau G}, \theta) + \cdots\Big)^2 + \cdots \qquad (7\text{-}13)$$

φ_{1x} is obtained by differentiating (5-9), and the terms representing the radial velocity component are replaced by the right-hand side of the boundary condition, in the form given by Eq. (7-5). After all terms have been expanded in Taylor series about $h^* = F$,

$$C_p = -2\Big(\delta^2 S_1'\big[\log \delta^2 F + \tfrac{1}{2}\tau G - \tfrac{1}{4}\tau^2 G^2 + \cdots\big] + \delta^2 g_1' + \cdots$$
$$+ \tau\delta^2\big[\varphi^*_{1x}(x, F, \theta) + \tfrac{1}{2}\tau F G \varphi^*_{1xh^*}(x, F, \theta) + \cdots\big] + \tau^2\delta^2\big[\varphi^*_{2x}(x, F, \theta) + \cdots\big] + \cdots\Big)$$
$$-\Big(\frac{\delta}{F}\big[FF' + \tfrac{1}{2}\tau(F^2 G)_x\big]\big[1 - \tfrac{1}{2}\tau G + \tfrac{3}{8}\tau^2 G^2 + \cdots\big]\big[1 + \delta^2 \log\delta\, 2S_1' + \cdots\big]$$
$$+ \frac{\tau}{\delta}\frac{G_\theta}{2F}\big[1 + \cdots\big]\big[\tau\delta^2 \varphi^*_{1\theta}(x, F, \theta) + \cdots\big]\Big)^2$$
$$\frac{1}{\delta^2 F^2}\big[1 + \cdots\big]\Big(\tau\delta^2 \varphi^*_{1\theta}(x, F, \theta) + \cdots\Big)^2 \qquad (7\text{-}14)$$

A considerable simplification occurs in the expression for drag. The drag coefficient is defined by a generalization of Eq. (6-11):

$$C_D = \frac{4}{\pi\delta^2}\int_0^1 dx \int_0^{2\pi} C_p \quad \delta F\sqrt{1+\tau G} \quad \frac{\partial}{\partial x}\left(\delta F\sqrt{1+\tau G}\right) d\theta$$

$$= \frac{4}{\pi}\int_0^1 dx \int_0^{2\pi} C_p \left(FF' + \frac{1}{2}\tau\left(F^2 G\right)_x\right) d\theta \tag{7-15}$$

Substituting for C_p and combining terms when possible,

$$\begin{aligned} C_D = C_{D_0} &+ \frac{2}{\pi}\tau\int_0^1 dx\int_0^{2\pi} C_{p_0}\left(F^2G\right)_x d\theta - \frac{4}{\pi}\tau\delta^2\int_0^1 dx\int_0^{2\pi}\left(F'^2\left(F^2G\right)_x + F'F''F^2G + 2FF'\varphi^*_{1_x}(x,F,\theta)\right)d\theta \\ &+ \frac{4}{\pi}\tau^2\delta^2\int_0^1 dx\int_0^{2\pi}\Big(-\frac{1}{2}FF'\left(3F'^2+FF''\right)G^2 - \frac{1}{2}F^2\left(4F'^2+FF''\right)GG_x \\ &\qquad -\frac{3}{4}F^3F'G_x^2 - \left(F^2G\right)_x\varphi^*_{1_x}(x,F,\theta) - F^2F'G\,\varphi^*_{1_{xh^*}}(x,F,\theta) \\ &\qquad -2FF'\varphi^*_{2_x}(x,F,\theta) - F'^2G_\theta\,\varphi^*_{1_\theta}(x,F,\theta) - \frac{F'}{F}\varphi^{*2}_{1_\theta}(x,F,\theta)\Big)d\theta \end{aligned} \tag{7-16}$$

where C_{p_0} and C_{D_0} are the results obtained for the body of revolution $h = \delta F(x)$.

It can now be argued that all terms linear in τ, including those neglected above which are of order $\tau\delta^4\log\delta$ and smaller, will drop out upon integration over θ. A more elaborate discussion is possible, but it seems sufficient to note that throughout the preceding development the Fourier expansions of terms linear in τ never contain a term which is independent of θ. It follows that there is no term in C_D which is linear in τ, and no generality has been lost by taking $\mu_2 = \tau^2\delta^2$

in Eq. (7-7a).

The last integral can be simplified by using relations of the form

$$\int_0^{2\pi} \mathcal{G}^2 \, d\theta = 2\pi \sum_{n=-\infty}^{\infty}{}' \mathcal{G}_n \mathcal{G}_{-n}$$

Each integral of a double sum may therefore be replaced by a single summation to be integrated over x . After rearranging terms, it is found that the remaining integration can be carried out explicitly to give

$$C_D = C_{D_0} + 2\tau^2\delta^2 \sum_{n=-\infty}^{\infty}{}' \left[\frac{1}{|n|} (F^2\mathcal{G}_n)'(F^2\mathcal{G}_{-n})' - F^2F'\mathcal{G}_n (3F'\mathcal{G}_{-n} + 2F\mathcal{G}_{-n}') \right]_{at\ x=1}$$

$$+ \; o(\tau^2\delta^2) \qquad (7\text{-}17)$$

where it has been assumed that the body has a pointed nose. It can be seen that the summation reduces to zero in three cases. For a body with a pointed base, $F(1) = 0$ and consequently all terms in the sum disappear. If instead the base is blunt, but the rate of change of shape with x is zero, then $F'(1) = \mathcal{G}_x(1,\theta) = 0$, and the sum again vanishes. Lastly, the sum can be zero if $\mathcal{G}(1,\theta) = \mathcal{G}_x(1,\theta) = 0$, i.e. if the base is blunt and circular and if the derivative of the shape perturbation $\mathcal{G}(x,\theta)$ is zero at the base.

For the three types of bodies described, therefore, the change in drag due to small deviations from a circular cross-section is of smaller order than $\tau^2\delta^2$:

$$C_D = C_{D_0} + o(\tau^2\delta^2) \qquad (7\text{-}18)$$

where δ represents the order of magnitude of the body thickness ratio, and τ the order of the perturbations in cross-section shape. This statement is actually a type of "area rule", since the non-circular body is compared to a body of revolution having everywhere the same cross-sectional area.

It is felt however that the analysis is not sufficiently general to allow application of the results to wing-body combinations. A separate expansion procedure is probably needed to give a good representation of a wing.

REFERENCES

1. von Karman, T.: The Similarity Law of Transonic Flow. J. Math. and Physics, Vol. 26, No. 3, 1947, pp. 182 ff.

2. Ward, G. N.: *Linearized Theory of Steady High-Speed Flow.* Cambridge University Press, 1955.

3. Liepmann and Puckett: Aerodynamics of a Compressible Fluid. John Wiley and Sons, 1947.

4. Sears, W. and Adams, M.: Slender-Body Theory. A Review and Extension. J. Aero. Sci. Vol. 20, 1953, p. 85.

5. Oswatitsch and Berndt: Aerodynamic Similarity of Axisymmetric Transonic Flow around Slender Bodies. KTH-Aero TN 15 (Sweden), 1950.

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12. Abstract:
The transonic approximation for flow over a slender body of revolution is discussed by means of a mathematical expansion procedure. The expansion is carried out in terms of a decreasing sequence of functions of the thickness ratio, and a similarity parameter relating the Mach number and thickness ratio is introduced. The first few terms of the expansions of the velocities near the axis are obtained, and similarity laws are derived for the pressure coefficient and drag coefficient of a body at zero incidence. By means of another expansion, an approximate "area rule" is obtained for a body of non-circular cross-section.